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Wave slam on a sphere penetrating a free surface*

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SUMMARY

The problem of vertical motion of a sphere across an oscillating free surface is analysed by assuming the fluid to be inviscid and the free surface to be an equipotential surface. New analytical solutions for the added-mass coefficients of a double spherical bowl are derived. These are used in the derivation of the drag coefficient of a sphere during vertical entry and of the slamming coefficient of a fixed sphere which is exposed to wave action. An additional important parameter in hydroballistics is the wetting factor of a sphere penetrating a free surface for which a new analytic solution is also derived in this paper. A comparison between some experimental data and the analytic expressions for the slamming coefficient and the wetting factor, shows good agreement between theory and measurements.

1. Introduction

The problem of water impact and water entry of bluff bodies striking a free-surface is a classical problem in naval hydrodynamics. The first analytical studies of this problem were preformed during the early thirties and were stimulated by an interest in the landing characteristics of seaplanes that were first designed at that time. Hence the classical works of von Kármán [27], who approximated the shape of the striking body by a growing flat plate, that of Wagner [28], who went a step beyond Kármán's analysis by considering also the water splash, and the work of Sedov [18], who introduced the powerful method of conformal transformations, are all considered as land-marks in the study of hydrodynamic impact and water entry. The second world war again served as an impetus for conducting further research in this field, primarily because of the interest in water entry and water exit of projectiles. In this category we make reference to the important contributions of Schiffman and Spencer [19, 20] who studied the case of a vertical entry of spheres and cones by using an approximate model which considers the flow about an expanding lens with its reflection in the original undisturbed surface. The work of Trilling [26] that followed Schiffman and Spencer's analysis seemed to have the greatest potential for practical application for arbitrary shaped bodies. Trilling's work may be also considered as a three-dimensional extension of Sedov's work in the sense that the immersed portion of the body is approximated by an equivalent prolate spheroid.

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About the same time the connection between ship slamming and hydroballistics was first investigated (Szebehely [22]) and numerous papers, applying analytical tools developed for hydrodynamic impact to the study of ship slamming, were published thereafter. Extensive general reviews on hydrodynamic impact and ship slamming are also available, and in particular we make reference to Szebehely [23], Chu and Abrahamson [3], Burt [2], Thigpen [25] and May [11].

In recent years a new application of the theory of hydrodynamical impact and water entry and exit was found in the field of ocean engineering and, in particular, in offshore technology. In designing offshore platforms in the open sea it is of primary importance to compute the wave-impact forces acting on the structure members. Here, in contrast with the water entry problem, the structure is fixed and the free surface is oscillating about the body such that the immersed portion of the body changes with time. The studies conducted so far in this problem dealt with the wave-slam on horizontal members of cylindrical cross-sections. (Kaplan and Silbert [9], Dalton and Nash [5], Faltinsen et al. [6], and Sarpkaya [17]). The analysis presented in these papers employs the added-mass concept of the wetted portion of the body together with its image within the free surface, and is based on the well-known solution for the added mass of a segment of a circle (Taylor [24]).

The motivation for the present study was an immediate interest of the offshore industries in estimating the slam forces acting on fixed spherical buoys that are exposed to severe wave action in the open sea. The analysis thus developed assumes the fluid to be inviscid and the free surface to be an equipotential surface, as discussed in the next section.

Since, to the best of the author's knowledge, an equivalent to Taylor's exact solution for the added mass of a segment of a cylinder is not available for a sphere, one of the important results of the present study is an analytical solution for the added-mass coefficient of a double spherical bowl. It is also verified that in the limit of maximum submergence the solution obtained for a double spherical bowl reduces to the one available for two touching spheres (Bentwich and Miloh [1]). The analysis also provides rather simple algebraic expressions for the wetting coefficient and for the splash contour caused by the vertical impact of the sphere, expressions that agree with available experimental measurements. Finally, the vertical slam force acting on the sphere is calculated for the general case in which both the free surface and the sphere are moving. The slamming coefficient and its variation with the submergence depth are also computed, demonstrating a major difference between the impact force acting on a sphere and on a cylinder. In spite of the close resemblance in the variation of the added-mass coefficients with submergence depth for a cylinder and for a sphere, it is found that the impact of a cylinder is of an impulsive nature whereas the slamming coefficient for a sphere is continuous, starting from zero at the instant of impact and attaining a maximum value immediately thereafter.

2. Statement of the problem

Consider the motion of a rigid sphere of radius R penetrating vertically a free surface with a time-dependent velocity $U(t)$. At the instant of contact between the sphere and the free surface we set $t = 0$ and the elevation of the undisturbed free surface (by the sphere) at this instant is denoted by $z = \zeta\ (x, y, t = 0) = \zeta_0\ (x, y)$. Here (x, y, z) denotes an inertial cartesian coordinate system with an origin at the mean free surface level, such that z is directed downward from the free surface in the direction of the gravitational acceleration g. If the characteristic wave length of the free surface disturbance is large compared with both the wave amplitude and the sphere radius R it is permissible to assume that in the immediate neighbourhood of the sphere, the free surface is horizontal and moves vertically with a time-dependent velocity $v_s(t)$ and that $\zeta = \zeta(r, t)$. For Stokes waves, for example, both the free surface velocity and elevation are harmonic in time t. Under the assumption of large wave-length the problem is symmetric about the z axis and thus can be formulated using a cylindrical coordinate system (z, r).

The problem of a free surface that is being struck by a solid body is generally characterized by small time scales and relatively large velocities which enable us to neglect viscous-drag forces and surface-tension forces in comparison with inertial hydrodynamic forces. Since the fluid is considered inviscid and the disturbance motion due to the penetrating sphere has started from rest, the flow is irrotational and may be derived from a velocity potential Φ. The total velocity potential $\Phi(z, r, t)$ is the sum of the undisturbed wave potential $\phi_s(r, t)$ and the disturbance potential $\phi(r, t)$ caused by the motion of the sphere through the free surface. Clearly, $\phi_s(z, r, t)$ is only locally axisymmetric and within linearized theory

$$v_s(t) = \frac{\partial \zeta}{\partial t}(0, t) = \frac{\partial \phi_s}{\partial z}(z, 0, t) \quad \text{on } z = \zeta(0, t) \cdot \tag{1}$$

The free surface boundary condition satisfied by both ϕ_s and ϕ is

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}(\nabla\phi \cdot \nabla\phi) + p/\rho = F(t) \quad \text{on } z = \zeta(r, t) + \zeta^*(r, t) \tag{2}$$

where p is the pressure, ρ the fluid density, $\zeta^*(r, t)$ is the free surface disturbance induced by the moving surface, and $F(t)$ is an arbitrary function of time. At large distances from the origin $(r \to \infty)$, both $\nabla\phi$ and $\partial\phi/\partial t$ vanish and the requirement that the pressure be zero on the free surface yields

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}(\nabla\phi \cdot \nabla\phi) = 0 \quad \text{on } z = \zeta(r, t) + \zeta^*(r, t). \tag{3}$$

The undisturbed wave potential ϕ_s satisfies the free surface boundary condition and is considered to be given. For the disturbance potential ϕ one may argue that the impact occurs during a very short time interval and that equation (3) may be linearized to give

$$\phi(\zeta + \zeta^*, r, t) = 0. \tag{4}$$

This free surface condition was first employed by Sedov [18] in his studies regarding the impact of two-dimensional solid bodies with a free surface. It is also known (Lamb, [10]) that during impact the dynamic pressure is proportional to the velocity potential which again verifies the linearized free surface condition (4). Equation (4) is also the large-frequency limit of the free surface boundary condition which governs the forced oscillation of a floating body on a free surface. Clearly, equation (4) does not imply the presence of any surface disturbance induced by the motion of the sphere. This may be physically justified for a small time interval (small penetration) after impact when gravitational forces may be neglected with respect to inertia forces. Equation (4), under the assumption of small free surface disturbance due to the free waves and the splash, may be applied on the horizontal interface,

$$\phi[\zeta(0,t)\, r,\, t] = 0. \tag{5}$$

When the fluid medium filled by the lower half-space $z > 0$ is quiescent before the sphere impact, the undisturbed free surface is stationary at $z = 0$ and hence, (5) reduces to

$$\phi(0,r,t) = 0. \tag{6}$$

Most previous studies on the impact of bodies on a free surface have ignored the free surface motion or the splash effect and used (6) as the corresponding free surface boundary condition, following the classical contributions of von Kármán [27], Sedov [18] and Trilling [26]. The first to consider the free surface disturbance caused by a body striking vertically on otherwise undisturbed free surface was Wagner [28], followed by the notable work of Schiffman and Spencer [20] on a cone in vertical impact. Schiffman and Spencer's theory takes into consideration the effect of piled-up water on the body so as to increase its wetted portion by assuming the free surface to be still a zero-potential surface moving with a finite vertical velocity. The so called splash effect is depicted schematically in Fig. 1. At the instant when the lowest point on the

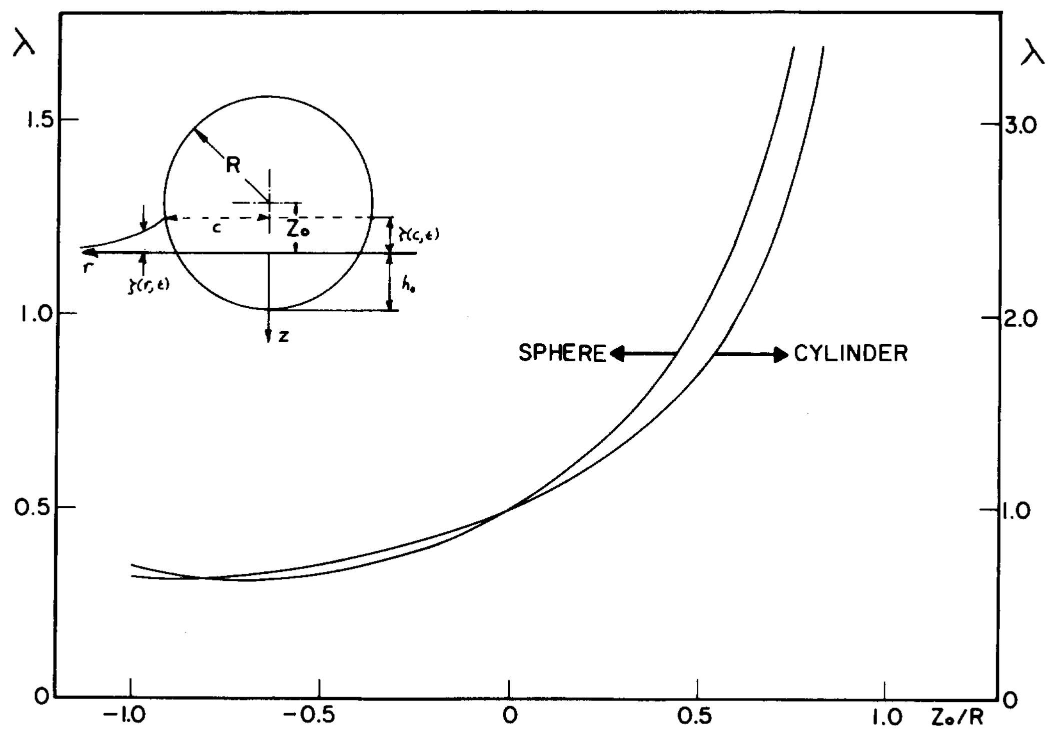

Figure 1. Variation of the added-mass coefficient of reflected cylindrical and spherical segments with the penetration depth (eqs. (38) and (42)).

body is at a depth h_0 below the original flat surface, the actual free surface contour is denoted by $\zeta(r, t)$ where, according to linear theory

$$\frac{\partial \zeta(r,t)}{\partial t} = \frac{\partial \phi(r,t)}{\partial z} \tag{7}$$

which is to be applied on the original flat surface together with (6). The deformed free surface intersects the sphere at $r = c$ at which point the free surface attains its maximum value $\zeta(c, t)$. An important parameter in the determination of the hydrodynamical reactions acting on a solid body striking a free surface is the so-called wetting factor defined by

$$C_w = 1 + \frac{\zeta(c,t)}{h_0}, \tag{8}$$

and represents the ratio between the elevations of the actual and the undisturbed free surface above the lowest point of the body.

The problem that we wish to solve is that of a sphere striking vertically an oscillating planar free surface with velocity $U(t)$ such that the instantaneous elevation of the free surface above the mean sea level is given by

$$\zeta(t) = A \sin\left(\frac{2\pi t}{T} + \varphi_s\right) \tag{9}$$

where A and T represent the amplitude and period of the free surface oscillation and φ_s is a phase shift defined by

$$\varphi_s = \sin^{-1}(\zeta_0/A) \tag{10}$$

where ζ_0 is the elevation of the free surface above the mean-sea-level at the instant of contact $t = 0$.

The undisturbed oscillating free surface is taken to be an equipotential plane on which (5) holds and the splash disturbance due to the penetration of the sphere is given by (7) applied on the undisturbed free surface $z = \zeta(t)$. These two boundary conditions, which must hold on the free surface are supplemented by the requirement that the sphere be an impermeable surface and that the disturbance velocity potential decays to zero at large distances from the sphere.

3. The Stokes stream function for the flow about a spherical bowl

The fact that the velocity potential vanishes on the planar free surface implies that the axisymmetric potential flow about the submerged portion of the sphere may be determined by considering the flow about the double-model which consists of the original submerged sphere portion and its mirror image in the planar free surface. Hence, the present study involves the solution for the axisymmetric potential flow of a double-spherical bowl moving with a time dependent

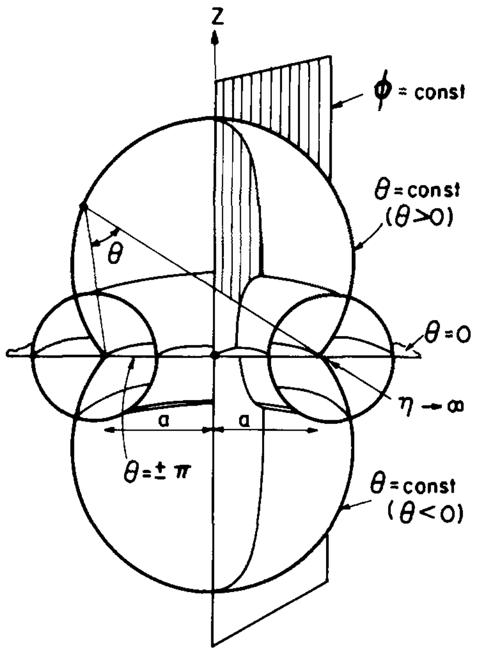

Figure 2. Definition sketch for toroidal coordinates.

velocity along its axis in an infinite expanse of fluid. The double-spherical bowl has two characteristic lengths, namely the sphere radius R and the instantaneous location of the sphere origin below the free surface Z_0 as depicted in Fig. 1.

A convenient triply-orthogonal coordinate system suitable for the present geometry is the toroidal coordinate system (η, θ, φ), where, following Moon and Spencer [14], η = const $(0 \leqslant \eta < \infty)$ are toroidal surfaces, θ = const $(-\pi \leqslant \theta < \pi)$ are spherical bowls, and φ = const $(0 \leqslant \varphi < 2\pi)$ are half-planes (Fig. 2). The orthogonal transformation between the cylindrical coordinate system (z', r'), with an origin on the instantaneous actual free surface, and the toroidal system is

$$z' = \frac{a \sin\theta}{\cosh\eta - \cos\theta} \quad ; \quad r' = \frac{a \sinh\eta}{\cosh\eta - \cos\theta} \tag{11}$$

where a is a characteristic parameter which is equal to the radius of the contour of intersection of the sphere and the free surface. The metric coefficients of the above transformation are

$$h_\eta = h_\theta = \frac{a}{\cosh\eta - \cos\theta} \quad ; \quad h_\varphi = \frac{a \sinh\eta}{\cosh\eta - \cos\theta}. \tag{12}$$

The equation of the double-spherical bowl $\theta = \theta_0$ = const is

$$r^2 + (z - a\cot\theta_0)^2 = (a/\sin\theta_0)^2 \tag{13}$$

which implies that the sphere radius R and the location of the sphere center below the free surface Z_0, are given by

$$R = a/\sin\theta_0 \;\; ; \;\; Z_0 = a\cot\theta_0 = R\cos\theta_0. \tag{14}$$

Since the family η = constant represents toroidal surfaces, it seems that the same toroidal (or ring) harmonics which have been used by Sternberg and Sadowsky [21] and by Miloh et al. [13] in analysing the three-dimensional motion of a finite torus, would be also appropriate for the present case. The only difference is that instead of applying the boundary conditions of no flow across the η = constant surface, one should apply them on the surface θ = constant, representing a spherical bowl. However, this procedure turned out to be inappropriate for the present case for the reasons explained below.

A typical exterior harmonic which is a normal separable solution of the Laplace equation in toroidal coordinates which vanishes at infinity ($\eta = 0$) is

$$\phi_e(\eta,\theta,\varphi) = (\cosh\eta - \cos\theta)^{1/2} P^m_{n-1/2}(\cosh\eta)\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix}\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}. \tag{15}$$

Similarly, a typical interior harmonic which is regular at the origin ($\eta \to \infty$) is

$$\phi_i(\eta,\theta,\varphi) = (\cosh\eta - \cos\theta)^{1/2} Q^m_{n-1/2}(\cosh\eta)\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix}\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix} \tag{16}$$

where n and m are integers and $P_{n-1/2}$ and $Q_{n-1/2}$ denote the Legendre functions of first and second kind, respectively.

These interior and exterior harmonics, while being applicable for flow-field description in the interior and exterior of a toroidal surface, are inappropriate for the corresponding flow description about a spherical bowl. The reason is that both in the interior and in the exterior of the spherical bowl η varies continuously from 0 to ∞, implying that on the bowl surface $\theta = \theta_o$ both the interior and the exterior harmonics are unbounded and can not be used as such for the present geometry.

This difficulty may be overcome by using the conal functions (Kegelfunktionen) first introduced by Mehler (1868), instead of the toroidal functions. The conal functions, which are basically Legendre functions of complex order, are discussed in Hobson [8] and were employed by Miloh [12] in analysing the blockage problem of a central body in a conical duct.

In the present formulation, it was found advantageous to employ the Stokes stream function instead of the velocity potential because of the axial symmetry of the flow, as demonstrated by Bentwich and Miloh [1] in the case of axisymmetrical flow about contiguous two-spheres. The Stokes stream function ψ is related to the velocity components in the (η, θ) directions by

$$u_\theta = \frac{1}{h_\eta r}\frac{\partial\psi}{\partial\eta} \;\; ; \;\; u_\eta = -\frac{1}{h_\theta r}\frac{\partial\psi}{\partial\theta}. \tag{17}$$

Equation (17) implies that continuity is satisfied and the additional requirement of irrotationality yields

$$\left(\frac{\partial}{\partial\eta}\frac{1}{r}\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\psi(\eta,\theta)=0. \tag{18}$$

To solve equation (18) for the stream function, we postulate a solution for ψ in the following form:

$$\psi(\eta,\theta)=a^2(\cosh\eta-\cos\theta)^{-1/2}H(\eta)\,\Theta(\theta) \tag{19}$$

which, when substituted into (18), yields

$$\left[\sinh\eta\frac{d}{d\eta}\frac{1}{\sinh\eta}\frac{d}{d\eta}+(p^2+1/4)\right]H(\eta)=0 \tag{20}$$

and

$$\left(\frac{d^2}{d\theta^2}-p^2\right)\Theta(\theta)=0 \tag{21}$$

where p is an arbitrary real number, not necessarily an integer.

A particular solution of (20) is

$$H(\eta)=\sinh\eta\frac{d}{d\eta}P_{-1/2+ip}(\cosh\eta)=\sinh\eta\frac{d}{d\eta}K_p(\cosh\eta) \tag{22}$$

where $K_p(\cosh\eta)$ denotes the conal function defined in (22) in terms of the Legendre function of the first kind of complex order. A general solution for the Stokes stream function is thus given by

$$\psi(\eta,\theta)=a^2(\cosh\eta-\cos\theta)^{-1/2}\sinh\eta\int_0^\infty A(p)\frac{d}{d\eta}K_p(\cosh\eta)\cosh p\theta\,dp \tag{23}$$

where $A(p)$ is an arbitrary function of p to be determined. It should be noted that the expression on the right-hand side of (23) is bounded for $0\leqslant\eta<\infty$ and hence may be used to describe the flow field exterior to the spherical bowl. The fact that the expression for the stream function is bounded for $\eta\to\infty$ has been demonstrated by Hobson [8] and Robin [16] and may be shown by considering the asymptotic expansion of $K_p(\cosh\eta)$ for large η reproduced in Appendix B.

The unknown function $A(p)$ is found by employing the boundary condition that

$$\psi(\eta,\pm\theta_0)=\tfrac{1}{2}r^2 \tag{24}$$

which implies that the double spherical bowl is a stream surface when moving with a unit velocity in the z direction. Substituting (11) and (23) in (24) yields the following integral equation for $A(p)$

$$\frac{d}{d\eta}\left\{\int_0^\infty A(p)K_p(\cosh\eta)\cosh(p\theta_0)dp+(\cosh\eta-\cos\theta_0)^{-1/2}\right\}=0. \tag{25}$$

To solve (25) for $A(p)$, it is necessary to obtain an integral representation in terms of the conal function of the non-integral part of (25). Such an expression is derived in Appendix A and here we reproduce only the final result:

$$(\cosh \eta - \cos \theta)^{-1/2} = \sqrt{2} \int_0^\infty \operatorname{sech}\,(p\pi) \cosh\,[p(\pi - \theta)] K_p(\cosh \eta) dp\ . \tag{26}$$

The fact that (25) is valid for any η implies, when using (26), that

$$A(p) = -\frac{\sqrt{2} \cosh\,[p(\pi - \theta_0)]}{\cosh(p\pi) \cosh(p\theta_0)} \tag{27}$$

or, after substitution in (23),

$$\psi(\eta, \theta) = -\sqrt{2} a^2 (\cosh \eta - \cos \theta)^{-1/2} \sinh \eta \int_0^\infty \frac{d}{d\eta} K_p(\cosh \eta) \tag{28}$$

$$\times \frac{\cosh\,[p(\pi - \theta_0)] \cosh(p\theta)}{\cosh(p\pi) \cosh(p\theta_0)}\, dp.$$

This disturbance stream function, which satisfies (24), vanishes at large distances from the sphere ($\eta \to 0$). Since ψ is an even function of θ its normal derivative vanishes on the undisturbed free surface $\theta = 0$ and hence the velocity potential is zero on this surface.

4. The added-mass of a spherical bowl

We consider the kinetic energy of the infinite expanse of fluid exterior to the double spherical bowl given by

$$T(\theta_0) = \tfrac{1}{2} \rho \iiint_{\forall} \left[\left(\frac{1}{h_\eta r} \frac{\partial \psi}{\partial \eta} \right)^2 + \left(\frac{1}{h_\theta r} \frac{\partial \psi}{\partial \theta} \right)^2 \right] h_\eta h_\theta r d\eta d\theta d\varphi. \tag{29}$$

This can be reduced, following Bentwich and Miloh [1], to a simpler form:

$$T(\theta_0) = \tfrac{1}{2} \pi \rho \int_0^\infty (r \frac{\partial \psi}{\partial \theta}) d\eta \tag{30}$$

which is to be evaluated at $\theta = \theta_0$.

Substitution of (11) and (28) into (30) yields,

$$\frac{\sqrt{2}\, T(\theta_0)}{\pi \rho a^3} = \int_0^\infty p\chi_1(p, \theta) \frac{\cosh\,[p(\pi - \theta_0)] \sinh(p\theta)}{\cosh(p\pi) \cosh(p\theta_0)} dp \,\Big|_{\theta = \theta_0} \tag{31}$$

$$+ \int_0^\infty \chi_2(p, \theta) \frac{\cosh\,[p(\pi - \theta_0)] \sinh(p\theta)}{\cosh(p\pi) \cosh(p\theta_0)} dp \,\Big|_{\theta = \theta_0}$$

where

$$\chi_1(p,\theta) = -\int_0^\infty \sinh^2\eta(\cosh\eta - \cos\theta)^{-3/2} \frac{d}{d\eta} K_p(\cosh\eta)d\eta \tag{32}$$

and

$$\chi_2(p,\theta) = \tfrac{1}{2}\int_0^\infty \sinh^2\eta(\cosh\eta - \cos\theta)^{-5/2} \sin\theta \frac{d}{d\eta} K_p(\cosh\eta)d\eta \tag{33}$$

$$= \tfrac{1}{3} \frac{\partial\chi_1(p,\theta)}{\partial\theta}$$

By employing the Fourier-Mehler integral transform it is possible to obtain an analytic expression for both $\chi_1(p,\theta)$ and $\chi_2(p,\theta)$. The derivation is given in Appendix B and here again we give only the final results:

$$\chi_2(p,\theta) = \sqrt{2}(2/p)(p^2 + 1/4)\,\mathrm{cosech}(p\pi)\cosh[p(\pi-\theta)], \tag{34}$$

$$\chi_2(p,\theta) = -\sqrt{2}(2/3)(p^2 + 1/4)\,\mathrm{cosech}(p\pi)\sinh[p(\pi-\theta)]. \tag{35}$$

Finally, substituting (34) and (35) into (31) and evaluating the resulting expression at $\theta = \theta_0$, we obtain the following expression for the kinetic energy:

$$T(\theta_0) = \pi\rho a^3 \int_0^\infty (4p^2 + 1)\left[\frac{\cosh^2[p(\pi-\theta_0)]\sinh(p\theta_0)}{\sinh(2p\pi)\cosh(p\theta_0)} - \frac{\sinh[2p(\pi-\theta_0)]}{6\sinh(2p\pi)}\right]dp. \tag{36}$$

It is clear that (36) converges for $0 \leqslant \theta_0 < \pi$ and that $T(\pi) = 0$. It is more convenient for numerical evaluation purposes to express (36) in a slightly different manner:

$$T(\theta_0) = \pi\rho R^3\left(\frac{\sin\theta_0}{\theta_0}\right)^3 \int_0^\infty (4\gamma^2 + \theta_0^2)\left\{\frac{\cosh^2[\gamma(\delta-1)]\sinh\gamma}{\sinh(2\gamma\delta)\cosh\gamma} - \frac{\sinh[2\gamma(\delta-1)]}{6\sinh(2\gamma\delta)}\right\}d\gamma \tag{37}$$

where $\delta = \pi/\theta_0$ and $\gamma = p\theta_0$.

To find the added-mass coefficient of the double-spherical bowl $\theta = \theta_0$ we divide the kinetic energy by the displaced mass of the fluid $\rho\forall(\theta_0)$ which renders for $\pi > \theta_0 > 0$

$$\lambda(\theta_0) = \frac{2T(\theta_0)}{\rho\forall(\theta_0)} = \frac{6T(\theta_0)}{\pi\rho R^3(2 + 3\cos\theta_0 - \cos^3\theta_0)} \tag{38}$$

Three limiting cases of (38), for which an exact solution is available, will next be discussed. The first case is $\theta_0 = \pi/2$ which corresponds to a sphere which is exactly half submerged, and hence the double body is a complete sphere. Substituting $\delta = 2$ in (37) and (38) and recalling that (Gradshteyn and Ryzhik, [7], p. 344)

$$\int_0^\infty \frac{4\gamma^2 + \pi^2/4}{\cosh(2\gamma)}d\gamma = \left(\frac{\pi}{2}\right)^3 \tag{39}$$

yields $\lambda(\pi/2) = 1/2$, a well known result for the added-mass coefficient of a complete sphere.

The second limiting case to be examined is that of a uniform axisymmetric flow past two spheres in contact, which, in the present coordinate system, corresponds to $\theta_0 = 0$. Taking the limit of (37) as $\theta_0 \to 0$ and $\delta \to \infty$ we get for the kinetic energy

$$T(0) = 2\pi\rho R^3 \int_0^\infty \gamma^2 \left[\frac{e^{-\gamma}}{\cosh\gamma} - \frac{4}{3} e^{-2\gamma} \right] d\gamma \tag{40}$$

which, following Gradshteyn and Ryzhik, [7], p. 361, may be evaluated as

$$T(0) = \tfrac{2}{3}\pi\rho R^3 \left[\tfrac{9}{8}\zeta(3) - 1\right], \tag{41}$$

where $\zeta(3)$ denotes the Riemann Zeta function. Since $\forall(0) = (4/3)\pi R^3$ the term in the parenthesis of (41) is in fact the added-mass coefficient of a flow past adjacent two spheres, which was also previously obtained by Bentwich and Miloh [1]. The same coefficient also represents the added-mass coefficient in a pure oscillatory heave motion below a free surface in the limit of infinitely large frequency and at the instant of contact with the free surface. The numerical value of this coefficient is $\lambda(0) = 0.3552314...$

The third limiting case of interest is just prior to contact with the free surface which corresponds to $\theta_o = \pi$. Clearly, for this case both the kinetic energy and the submerged volume of the sphere vanish. However, closer scrutiny of the ratio between the two reveals that $\lambda(\pi) \to \infty$. This result should be also compared against the classical solution of Taylor [24] for the added-mass coefficient of a double cylindrical segment which also yields infinitely large added-mass coefficient for the cylinder at the instant of contact. Taylor's solution for the added-mass coefficient of a segment of a circle of radius r and submergence z is

$$\lambda(\varphi) = \left[\frac{2\pi^3(1-\cos\varphi)}{3(2\pi-\varphi)^2(\varphi-\sin\varphi)} + \frac{\pi(1-\cos\varphi)}{3(\varphi-\sin\varphi)} - 1 \right] \tag{42}$$

where φ is the angular opening of the segment given by

$$\varphi = 2\cos^{-1}(1 - z/r). \tag{43}$$

This yields

$$\lambda(\pi) = 1 \quad \text{and} \quad \lambda(2\pi) = \frac{\pi^2}{6} - 1 \tag{44}$$

The added-mass coefficients for both a sphere and a cylinder as a function of the submergence are plotted in Fig. 1. Noteworthy is the close resemblance between the two.

5. Determination of the splash contour

The linearized solution for the actual free surface profile due to the penetration of the sphere is given by (7) and is expressed here in terms of the Stokes stream function using toroidal coordinates:

$$\frac{\partial\zeta(\eta,t)}{\partial t} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial\eta}{\partial z} - \frac{1}{r}\frac{\partial\psi}{\partial\eta}\frac{\partial\theta}{\partial z}\Bigg|_{\theta=0} \tag{45}$$

where $\theta = 0$ represents the undisturbed free surface ($z = 0$). Since the transformation $(z, r) \leftrightarrow (\eta, \theta)$ is orthogonal, we obtain from (11) and (12)

$$\frac{\partial\theta}{\partial z} = \frac{1}{h_\theta^2}\frac{\partial z}{\partial\theta} = \frac{1}{a}(\cosh\eta\cos\theta - 1), \tag{46}$$

$$\frac{\partial\eta}{\partial z} = \frac{1}{h_\eta^2}\frac{\partial z}{\partial\eta} = -\frac{1}{a}\sinh\eta\sin\theta$$

which, when evaluated on the undisturbed free surface, yields

$$\frac{\partial\theta}{\partial z} = \frac{1}{a}(\cosh\eta - 1), \quad \frac{\partial\eta}{\partial z} = 0. \tag{47}$$

Hence, (45) reduces for $\theta = 0$ to

$$\frac{\partial\zeta(\eta,t)}{\partial t} = -\frac{1}{r}\frac{\partial\theta}{\partial z}\frac{\partial\psi}{\partial\eta} = -\frac{1}{a^2}\frac{(\cosh\eta - 1)^2}{\sinh\eta}\frac{\partial\psi}{\partial\eta}\Bigg|_{\theta=0}, \tag{48}$$

or

$$\zeta(\eta,t) = -\frac{(\cosh\eta - 1)^2}{\sinh\eta}\int_0^t \frac{1}{a^2}\frac{\partial}{\partial\eta}\psi(\eta,t)dt. \tag{49}$$

Denoting the instantaneous location of the sphere center below the undisturbed free surface by $Z_0(t)$, we have from (13)

$$dZ_0(t) = V_r(t)dt = -R\sin\theta_0(t)d\theta_0 \tag{50}$$

where $V_r(t)$ denotes the relative velocity between the moving sphere and the oscillating free surface. Substitution of (28) and (50) in (49) gives

$$\zeta(\eta,t) = \frac{\sqrt{2}R(\cosh\eta - 1)^2}{\sinh\eta}\frac{\partial}{\partial\eta}\int_{\theta_0(t)}^{\pi}\frac{\sin\theta_0\sinh\eta}{(\cosh\eta - 1)^{1/2}}\int_0^\infty \frac{d}{d\eta}K_p(\cosh\eta) \tag{51}$$

$$\times\frac{\cosh[p(\pi - \theta_0)]}{\cosh(p\pi)\cosh(p\theta_0)}\,dp\,d\theta_0$$

since the stream function was originally defined for a unit axial velocity.

Since $\eta \to \infty$ at the intersection of the sphere with the undisturbed free surface, the splash contour in the neighbourhood of the sphere may be determined by examining the asymptotic behaviour of (51) for large values of η. The corresponding asymptotic expansion of the conal function $K_p(\cosh\eta)$ is, following Robin ([16], Vol. 3, p. 153),

$$K_p(\cosh \eta) \sim \left(\frac{2}{\pi e^\eta}\right)^{1/2} \left(\frac{\cot p\pi}{p}\right)^{1/2} \cos(p\eta + \tilde{\delta})\,[1 + O(e^{-2\eta})] \tag{52}$$

where $\tilde{\delta}$ is some phase angle.

For large values of η the argument of the trigonometric function implies that most of the contribution to the integral in (51) is from small values of p. This is also evident when the Kelvin's method of stationary phase is used, and hence equation (51) is approximated by

$$\zeta(\eta, t) = \frac{\sqrt{2}R(\cosh \eta - 1)^2}{\sinh \eta} \frac{\partial}{\partial \eta} \int_{\theta_0}^{\pi} \frac{\sin \theta_0 \sinh \eta}{(\cosh \eta - 1)^{1/2}} \int_0^\infty \frac{d}{d\eta} K_p(\cosh \eta) \tag{53}$$

$$\times \frac{\cosh[p(\pi - \theta_0)]}{\cosh(p\pi)}\, dp d\theta_0$$

which may be also considered as an upper bound for the actual splash contour. Substitution of (26) in (53) yields

$$\zeta(\eta, t) = \frac{R(\cosh \eta - 1)^2}{\sinh \eta} \frac{\partial}{\partial \eta} \left[\frac{\sinh^2 \eta}{(\cosh \eta - 1)^{1/2} (\cosh \eta - \cos \theta_0)^{1/2}} \right] \Bigg|_{\theta_0}^{\pi}$$

$$= -\frac{R}{2} (\cosh \eta - 1)^{1/2} \int_{\theta_0}^{\pi} \sin \theta_0 (\cosh \eta - \cos \theta_0)^{-5/2} [(1 + \cos \theta_0) \tag{54}$$

$$\times (2 \cosh \eta - \tfrac{3}{2} \cosh^2 \eta - \tfrac{1}{2}) + \sinh^2 \eta] d\theta_0$$

which in the neighbourhood of the sphere were $\cosh \eta >> \cos \theta_0$, gives

$$\zeta(\eta, t) = \frac{R}{2} \frac{(\cosh \eta - 1)^{1/2}}{(\cosh \eta - \cos \theta_0)^{5/2}} (1 + \cos \theta_0) [(2 \cosh \eta - \tfrac{1}{2} \cosh^2 \eta - \tfrac{3}{2}) -$$

$$- \tfrac{1}{2}(1 - \cos \theta_0)(2 \cosh \eta - \tfrac{3}{2} \cosh^2 \eta - \tfrac{1}{2})]. \tag{55}$$

In particular the maximum height of the splash contour is at the intersection of the sphere and the free surface, namely at $\eta \to \infty$ for which equation (55) gives

$$\zeta(\infty, t) = -\frac{R}{4} (1 + \cos \theta_0) [1 - \tfrac{3}{2}(1 - \cos \theta_0)]. \tag{56}$$

The wetting factor, defined in (8) may be found by substituting (56) in (8) and letting $h_0 = R(1 + \cos \theta_0)$. Hence we get a rather simple expression:

$$C_w(\theta_0) = \tfrac{3}{2} [1 - \tfrac{1}{4}(1 + \cos \theta_0)] = \tfrac{3}{2}(1 - \frac{h_0}{4R}). \tag{57}$$

Equation (57) shows that at the instant of impact the wetting factor attains its maximum value of 3/2 and decreases monotonically with a further increase in the penetration depth. Schiffman

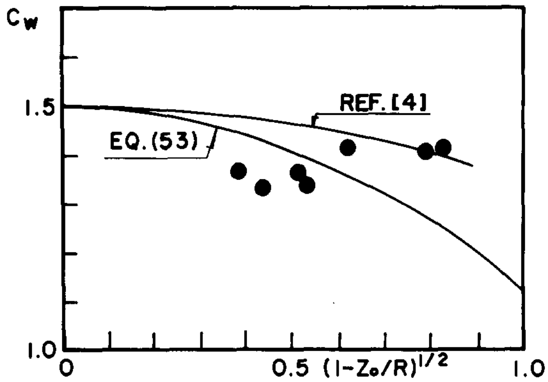

Figure 3. A comparison between analytical solution for the variation of the wetting factor versus the penetration depth (eq. (53)) with the empirical result of Cooper [4] and the experimental data given in Burt [2].

and Spencer's [20] analysis, which is based on Wagner's [28] expanding disk model yields $C_W = 3/2$ for all penetrations. This theoretical value was obtained by using a well-known expression for the free surface disturbance caused by a vertical impact of a disk,

$$\frac{d\zeta}{dt} = V\left[\frac{c}{(r^2 - c^2)^{1/2}} - \sin^{-1}\left(\frac{c}{R}\right)\right], \tag{58}$$

where V is the impact velocity, c is the radius of the disk and r is the radial distance in the plane of the disk.

It should be noted however that actual measurements of splash contours (White [32], Nisewanger [15], Burt [2]) clearly demonstrate that the wetting factor changes considerably with the penetration depth as predicted by the theoretical model (57). A comparison between the experimental measurements of White [32] with the theoretical model (57) is presented in Fig. 3 and the agreement is satisfactory. Also given in the same figure is the semi-empirical solution due to Cooper [4] which is in fact an improvement of Schiffman and Spencer's [19] expanding disk model when higher order terms are taken into account.

6. The vertical force experienced by the sphere

To find the vertical hydrodynamic force acting on a sphere penetrating a moving free surface we employ Lagrange's equation of motion. Let the instantaneous free surface elevation above the mean water level be denoted by $\zeta(t)$, and the time derivative $\dot{\zeta}(t)$ represents the vertical velocity of the free surface. The instantaneous elevation of the sphere center above the mean water level is denoted by z such that $\dot{z} = -U(t)$, where $U(t)$ is the time-dependent vertical velocity of the sphere. The total kinetic energy imparted to the fluid by the vertical motion of the sphere is (see (38))

$$\tau = \tfrac{1}{2}\rho\forall(\theta_0)\,\lambda(\theta_0)(U - \dot{\zeta})^2 = T(\theta_0)(U - \dot{\zeta})^2, \tag{59}$$

and the total vertical hydrodynamical (including buoyancy) force experienced by the sphere is therefore

$$F = -\frac{d}{dt}\,[T(\theta_0)(U - \dot{\zeta})] + \rho\forall(\theta_0)(g + \ddot{\zeta})$$

$$= -T(\theta_0)(\dot{U} - \ddot{\zeta}) + \frac{\partial T(\theta_0)}{\partial z}\,(U - \dot{\zeta})^2 + \rho\forall(\theta_0)(g + \ddot{\zeta}) \tag{60}$$

Thus, the hydrodynamical force acting on a sphere penetrating with constant velocity an otherwise calm free surface is

$$F = \frac{\partial T(\theta_0)}{\partial z}\,U^2 + \rho g\forall(\theta_0) \tag{61}$$

whereas the wave-slam force experienced by a stationary sphere in the presence of a moving free surface is given by

$$F = [\rho\forall(\theta_0) + T(\theta_0)]\,\ddot{\zeta} + \frac{\partial T(\theta_0)}{\partial z}\,\dot{\zeta} + \rho g\forall(\theta_0) \tag{62}$$

which is identical with the expression given by Kaplan and Silbert [9] for the wave slam on a cylindrical platform.

An important parameter in vertical water entry is the so-called slamming coefficient defined by

$$C_s(\theta_0) = \frac{2F}{\rho\pi R^2 U^2} = \frac{2}{\pi R^2}\,\frac{\partial}{\partial z}\,[\lambda(\theta_0)\forall(\theta_0)] = \frac{2}{\pi\rho R^2}\,\frac{\partial T(\theta_0)}{\partial z}. \tag{63}$$

The slamming coefficient of a sphere striking a free surface vertically may be then found by substituting (36) and (38) into (63), which yields

$$\begin{aligned} C_s(\theta_0) = &-3\sin(2\,\theta_0)\int_0^\infty (4p^2+1)\left\{\frac{\cosh^2[p(\pi-\theta_0)]\sinh(p\theta_0)}{\sinh(2p\pi)\cosh(p\theta_0)}\right. \\ &\left. -\frac{\sinh[2p(\pi-\theta_0)]}{6\sinh(2p\pi)}\right\}dp \\ &+2\sin^2\theta_0\int_0^\infty p(4p^2+1)\left\{\frac{\sinh[2p(\pi-\theta_0)]\sinh(p\theta_0)}{\sinh(2p\pi)\cosh(p\theta_0)}\right. \\ &\left. -\frac{\cosh^2[p(\pi-\theta_0)]}{\sinh(2p\pi)\cosh^2(p\theta_0)}+\frac{\cosh[2p(\pi-\theta_0)]}{3\sinh(2p\pi)}\right\}dp\,. \end{aligned} \tag{64}$$

The slamming coefficient of a sphere is depicted in Fig. 4 against the dimensionless submergence Z_0/R. This coefficient rises sharply from zero at first contact to a maximum value of $C_s \approx 0.96$ at

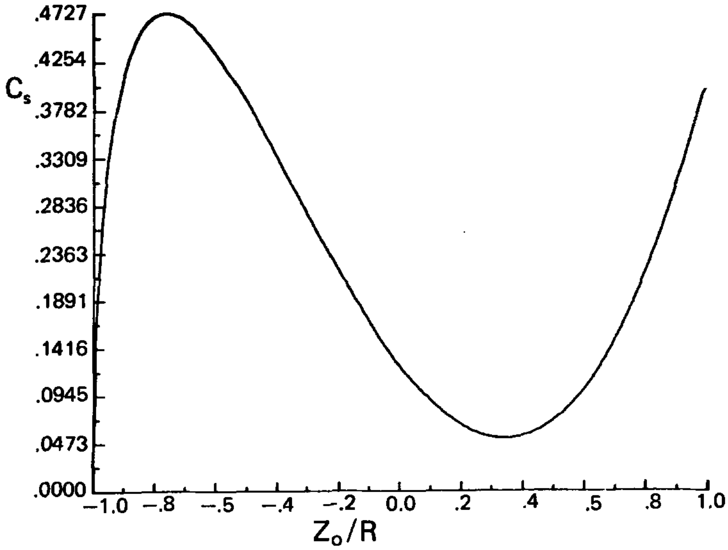

Figure 4. Variation of the slamming coefficient of a sphere versus the penetration depth as given by eq. (64).

$Z_0/R \sim -0.8$ and then it decreases to a minimum at $Z_0/R \sim 0.3$. Experimental measurements on the vertical entry of a sphere, conducted by Nisewanger [15] with a constant entry velocity of 23.5 fps and reported in Wardlaw et al. [29], show the maximum slamming coefficient as $C_s \sim 1.0$ in agreement with our theoretical prediction. This maximum value also agrees with impact-resistance measurements reported by Watanabe [30] for a sphere (28 cm in diameter) falling freely upon a water surface.

The corresponding slamming coefficient of a cylinder striking a free surface is given, in an analogous manner to (63), by

$$C_s(\varphi) = \frac{F}{\rho r U^2} = \frac{1}{r}\frac{\partial}{\partial z}[\lambda(\varphi)A(\varphi)] = \frac{1}{\rho r}\frac{\partial T(\varphi)}{\partial z} \tag{65}$$

where φ is defined in (43) and $A(\varphi)$ is the immersed area of the cylinder given by

$$A(\varphi) = \tfrac{1}{2}r^2(\varphi - \sin\varphi). \tag{66}$$

Substituting Taylor's solution (42) for the added mass of a cylinder, together with (21) and (66) into (65), we find

$$C_s(\varphi) = \operatorname{cosec}\left(\tfrac{1}{2}\varphi\right)\left\{\frac{2\pi^3}{3}\left[\frac{\sin\varphi}{(2\pi-\varphi)^2} + \frac{2(1-\cos\varphi)}{(2\pi-\varphi)^3}\right] + \frac{\pi}{3}\sin\varphi + \cos\varphi - 1\right\}. \tag{67}$$

The variation of the cylinder slamming coefficient with the submergence depth as given by eq. (67) and depicted in Fig. 5, implies that $C_s(0) = \pi$. Hence the impact force of a cylinder striking a free surface is of an impulsive nature and rises instantaneously from zero to π at the instant of

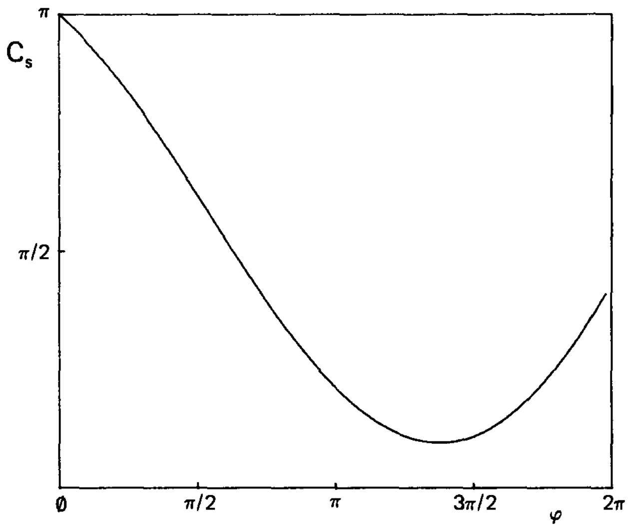

Figure 5. Variation of the slamming coefficient of an infinite cylinder, as given by eq. (67), versus the opening angle φ defined in eq. (43).

contact. It should be noted that initial stage of impact of a cylinder differs from that of a sphere where the slamming coefficient is zero at first contact. The fact that the slamming coefficient of a cylinder immediately after contact is indeed of an impulsive nature and agrees with the theoretical limit $C_s(0) = \pi$, was verified experimentally by Sarpkaya [17] by using a large U-tube tunnel with a free surface oscillating past fixed cylinders 3 to 8 inches in diameter. Apparently Watanabe [30, 31] was the first to verify experimentally that for small penetration depths the slamming coefficient of a sphere increases monotonically from zero to some maximum value, whereas that of a cylinder decreases monotonically from an initial maximum value.

Appendix A

A typical normal solution of the Laplace equation in toroidal coordinates is

$$\phi(\eta, \theta) = (\cosh \eta - \cos \theta)^{1/2} K_p(\cosh \eta) \begin{Bmatrix} \sinh p\theta \\ \cosh p\theta \end{Bmatrix}. \tag{A1}$$

Hence, since unity is also harmonic,

$$1 = (\cosh \eta - \cos \theta)^{1/2} \int_0^\infty C(p) K_p(\cosh \eta) \cosh [p(\theta - \alpha)] dp \tag{A2}$$

where $C(p)$ is an unknown function of p, and α is an undetermined constant. Equation (A2) is valid for any η and for the particular choice of $\eta = 0$ it reduces to

$$(1 - \cos \theta)^{-1/2} = \int_0^\infty C(p) \cosh [p(\theta - \alpha)] dp \tag{A3}$$

since, $K_p(1) = 1$ (Hobson [8], p. 455).

Equation (A3) may be also written as

$$(\sqrt{2}\cos(\tfrac{1}{2}\theta))^{-1} = \int_0^\infty C(p)\cosh[p(\theta-\alpha+\pi)]\,dp = \sqrt{2}\int_0^\infty \frac{\cosh(p\theta)}{\cosh(p\pi)}\,dp \tag{A4}$$

where the relation between the first and third function in (A4) is given by Gradshteyn and Ryzhik ([7], p. 344). Equation (A4) and analytic continuation yield

$$C(p) = \sqrt{2}\,\mathrm{sech}\,(p\pi) \quad ; \quad \alpha = \pi \tag{A5}$$

which when substituted into (A2) renders the desired expression,

$$(\cosh\eta - \cos\theta)^{-1/2} = \sqrt{2}\int_0^\infty \mathrm{sech}(p\pi)\cosh[p(\theta-\pi)]\,K_p(\cosh\eta)dp. \tag{A6}$$

Appendix B

The governing differential equation for the conal function is

$$\left[\frac{1}{\sinh\eta}\frac{d}{d\eta}\sinh\eta\frac{d}{d\eta} + (p^2+1/4)\right]K_p(\cosh\eta) = 0, \tag{B1}$$

from which it follows that

$$\sinh\eta\frac{d}{d\eta}K_p(\cosh\eta) = (p^2+1/4)\int_\eta^\infty K_p(\cosh q)\sinh q\,dq. \tag{B2}$$

The above relation makes it possible to integrate (23) by parts which results in

$$\chi_1(p,\theta) = 2(p^2+1/4)\int_0^\infty (\cosh\eta-\cos\theta)^{-1/2}K_p(\cosh\eta)\sinh\eta\,d\eta. \tag{B3}$$

In deriving (B3) a use has been also made of the fact that $K_p(\cos h\eta)$ is bounded for $\eta = 0$ and has the following asymptotic expansion for large η (Robin [16], Vol. 3, p. 153)

$$K_p(\cosh\eta) \sim \left(\frac{2}{\pi e^\eta}\right)^{1/2}\left(\frac{\cot p\pi}{p}\right)^{1/2}\cos(p\eta+\tilde{\delta})[1+O(e^{-2\eta})] \tag{B4}$$

where $\tilde{\delta}$ is some phase angle.

Next, we refer to a particular form of the Fourier-Mehler integral transformation for conal functions (Robin [16], Vol. 3, p. 165)

$$f(\cosh\eta) = \int_0^\infty p\,\mathrm{tgh}(p\pi)K_p(\cosh\eta)dp\int_0^\infty K_p(\cosh\eta')f(\cosh\eta')\sinh\eta'd\eta' \tag{B5}$$

which yields the following inversion formula for a pair of functions $f(\cosh\eta)$ and $g(\cosh\eta)$ satisfying the Riemann-Lebesgue lemma

$$f(\cosh \eta) = \int_0^\infty K_p(\cosh \eta) g(p)\, dp. \tag{B6}$$

$$g(p) = p \operatorname{tgh}(p\pi) \int_0^\infty K_p(\cosh \eta) f(\cosh \eta) \sinh \eta \, d\eta. \tag{B7}$$

These relations together with (A5) yield the desired expression for the function χ_1 defined in (B3)

$$\chi_1(p, \theta) = \sqrt{2}(2/p)(p^2 + ¼) \operatorname{cosech}(\rho\pi) \cosh[p(\pi - \theta)], \tag{B8}$$

and, because of (24),

$$\chi_2(p, \theta) = -\sqrt{2}(2/3)(p^2 + 1/4) \operatorname{cosech}(p\pi) \sinh[p(\pi - \theta)]. \tag{B9}$$

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Forthcoming papers

The following papers have been accepted for publication in the Journal of Engineering Mathematics:

1. Alternative integral representations for the Green function of the theory of ship wave resistance, by F. Noblesse.
2. Injection from a finite section of a flat plate placed parallel to a uniform stream, by F. T. Smith and S. C. R. Dennis.
3. Non-uniform slot injection into a laminar boundary layer, by N. Riley.
4. On the wave resistance of a submerged semi-elliptical body, by L. K. Forbes.
5. The transient heaving motion of floating cylinders, by R. W. Yeung.
6. Linear and nonlinear hyperbolic wave problems with input sets, by E. Adams and W. F. Ames.
7. Optimal design of two-dimensional leaflet valves, by E. O. Tuck.